Chapter 3
Analysis and Transmission of Signals

Instructor: Oluwayomi Adamo
Introduction

- Engineers view signals in terms of frequency spectra
- Audio signals having bandwidth of 20 kHz
- Loudspeakers responding to 20 kHz audio signal
- We will study spectra representation of aperiodic signals
Aperiodic Representation

- Aperiodic signal $g(t)$ can be represented as a continuous sum (integral) of everlasting exponential.
- We have to construct a new periodic $g_{T_0}(t)$ by repeating signal $g(t)$ every $T_0$ seconds.
- $g_{T_0}(t)$ can be represented as exponential Fourier series.
- As $T_0 \to \infty$:

Exponential Fourier series for $g_{T_0}(t)$

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \quad D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g_{T_0}(t)e^{-jn\omega_0 t} \, dt$$

$$\lim_{T_0 \to \infty} g_{T_0}(t) = g(t)$$
Aperiodic Representation

- Integrating \( gT0 \) over \((-T0/2, T0/2)\) is the same as integrating \( g(t) \)

\[
D_n = \frac{1}{T_0} \int_{-\infty}^{\infty} g(t)e^{-jn\omega_0 t} \, dt
\]

- If we define a function \( G(f) \) as a function \( \omega \)

\[
G(f) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t} \, dt = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft} \, dt
\]

\[
D_n = \frac{1}{T_0} G(nf_0)
\]

- Shows that Fourier Coefficient \( D_n \) are \((1/T0 \text{ times})\) the samples of \( G(f) \) uniformly spaced at intervals of \( f_0 \) Hz

- \((1/T0)(G(f))\) is the envelope for the coefficient \( D_n \)
Aperiodic Representation

• If $T_0$ is doubled, the envelop is halved and magnitude gets smaller as $T_0$ is doubled more.

• Shape remains the same

![Diagram showing an envelope and its components](image)

• $g_{T_0}(t)$ becomes

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} \frac{G(nf_0)}{T_0} e^{jn2\pi f_0t}$$

• Because $T_0 \to \infty$, $f_0 = 1/T_0$ becomes infinitesimal ($f_0 \to 0$). Hence $\Delta f = 1/T_0$

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} [G(n\Delta f)\Delta f] e^{(j2\pi n\Delta f)t}$$

• $g_{T_0}(t)$ can be expressed as a sum of everlasting exponentials of

$$0, \pm \Delta f, \pm 2\Delta f, \pm 3\Delta f \ldots \text{(fourierseries)}$$
• The amount of the component of frequency $n\Delta f$ is $G(n\Delta f)\Delta f$

• As $T_0 \to \infty$, $\Delta f \to 0$ and $g_{T_0}(t) \to g(t)$:

$$g(t) = \lim_{T_0 \to \infty} g_{T_0}(t) = \lim_{\Delta f \to 0} \sum_{-\infty}^{\infty} G(n\Delta f)e^{(j2\pi n\Delta f)t} \Delta f$$

• This is the area under $G(f)e^{j2\pi ft}$ called Fourier integral:

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft} \, dt$$
Fourier Transform

- Direct Fourier transform of g(t)
  
  \[ G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft} \, dt \]

- Inverse Fourier Transform of G(f)

  \[ g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft} \, dt \]

- Symbolically:

  \[ G(f) = F[g(t)] \]

  \[ g(t) = F^{-1}[G(f)] \]

- We can plot G(f) as a function of f and amplitude and angle (phase) spectra exist

  \[ G(\omega) = |G(\omega)|e^{j\theta(\omega)} \]
Conjugate Symmetry Property

• If $g(t)$ is a real function of $t$, $G(f)$ and $G(-f)$ are complex conjugate

\[ G(-f) = G^*(f) \]

• This means:

\[ |G(-f)| = |G(f)| \]
\[ \theta g(-f) = -\theta g(f) \]
Find the Fourier Transform of $e^{-at}u(t)$

- $G(\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t} \, dt = \int_{0}^{\infty} e^{-(a+j\omega)t} \, dt = \left. \frac{-1}{a+j\omega} e^{-(a+j\omega)t} \right|_{0}^{\infty}$

  $|e^{-j\omega t}| = 1$

  $t \to \infty, e^{-(a+j\omega)t} = e^{-at}e^{-j\omega t} = 0 \text{ if } a > 0$

  $G(\omega) = \frac{1}{a+j\omega} \quad a > 0$

- Expressing $a+j\omega$ in polar form $\sqrt{a^2 + \omega^2} e^{j\tan^{-1}(\frac{\omega}{a})}$

  $G(\omega) = \frac{1}{\sqrt{a^2 + \omega^2}} e^{-j\tan^{-1}(\frac{\omega}{a})}$

  $|G(\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}$ and $\theta_g(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$
• The amplitude spectrum $|G(f)|$ and the phase spectrum $\theta_g(f)$ is shown:
Existence of Fourier Transform

• Fourier transform exist for a function $g(t)$ if:

$$\int_{-\infty}^{\infty} |g(t)| \, dt < \infty$$

• The physical existence of a signal is a sufficient condition for the existence of its transform

• The fourier transform is linear if:

$$a_1 g_1(t) + a_2 g_2(t) \leftrightarrow a_1 G_1(\omega) + a_2 G_2(\omega)$$
Transform of Useful Functions

- Unit Rectangular Function

\[
\text{rect}(x) = \begin{cases} 
0 & |x| > \frac{1}{2} \\
\frac{1}{2} & |x| = \frac{1}{2} \\
1 & |x| < \frac{1}{2} 
\end{cases}
\]

- The unit rectangular pulse \( \text{rect}(x) \) in (a) is expanded by a factor \( \tau \) \( \text{rect}(x/\tau) \) in (b)
Unit Triangular function

- Triangular pulse $\Delta(x)$ of unit height and unit width, centered at the origin:

$$\Delta(x) = \begin{cases} 
0 & |x| > \frac{1}{2} \\
1 - 2|x| & |x| < \frac{1}{2} 
\end{cases}$$
Sinc function

- This is the function \( \frac{\sin x}{x} \) (sine over argument)
- Plays an important role in signal processing

\[
\text{sinc}(x) = \frac{\sin x}{x}
\]

- It is an even function of \( x \)
- \( \text{Sinc}(x) = 0 \) when \( \sin x = 0 \) except at \( x = 0 \), i.e. \( \text{sinc}(x) = 0 \) for \( t = \pm \pi, \pm 2\pi, \pm 3\pi \)
- \( \text{Sinc}(0) = 1 \)
- \( \text{Sinc}(x) \) is an oscillating function with decreasing amplitude.
- It has a unit peak at \( x = 0 \) and zero crossings at integer multiples of \( \pi \)
- Product of \( \sin x \) (\( T_0 = 2\pi \)) and \( 1/x \)
Find the Fourier transform of \( g(t) = \Pi(t/\tau) \)

**Solution**

\[
G(f) = \int_{-\infty}^{\infty} \Pi\left(\frac{t}{\tau}\right) e^{-j2\pi ft} \, dt
\]

Since \( \Pi(t/\tau) = 1 \) for \( |t| < \tau/2 \), and since it is zero for \( |t| > \tau/2 \)

\[
G(f) = \int_{-\tau/2}^{\tau/2} e^{-j2\pi ft} \, dt
\]

\[
= -\frac{1}{j2\pi f} (e^{-j\pi f\tau} - e^{j\pi f\tau}) = \frac{2\sin(\pi f\tau)}{2\pi f}
\]

\[
= \tau \frac{\sin(\pi f\tau)}{\pi f} = \tau \sin c(\pi f\tau)
\]

\[
\Pi\left(\frac{t}{\tau}\right) \leftrightarrow \tau \sin c\left(\frac{\omega \tau}{2}\right) = \tau \sin c(\pi f\tau)
\]
• Since sinc(x) = 0 when x = ±nπ
• Sinc(ωt/2) = 0 when ωt/2 = ±nπ when f = ±n/τ (n = 1, 2, 3, ……)
• G(f) is real, hence spectra is just a single plot of G(f)
• See plot in the text
• The spectrum peaks at f = 0 and decays at higher frequencies.
• Π(t/τ) is a low-pass signal with most of the signal energy in lower frequency components
• Signal bandwidth – difference between highest (significant) frequency and lowest (significant) frequency in the signal spectrum
• A rough estimate of bandwidth of a rectangular pulse is 2π/τ rad/s or 1/τ Hz
Find the Fourier transform of the unit impulse signal \((\delta(t))\)

- Using sampling property of impulse function

\[
F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft} \, dt = e^{-j2\pi f \cdot 0} = 1
\]
Find the inverse Fourier transform of \( \delta(2\pi f) = \frac{1}{2\pi} \delta(f) \)

- Using sampling property of impulse function (pg. 33)

\[
F^{-1}[\delta(2\pi f)] = \int_{-\infty}^{\infty} \delta(2\pi f) e^{j2\pi ft} df = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(2\pi f) e^{j2\pi ft} d(2\pi f)
\]

\[
= \frac{1}{2\pi} e^{-j0t} = \frac{1}{2\pi}
\]

\[
\frac{1}{2\pi} \Leftrightarrow \delta(2\pi f)
\]

1 \(\Leftrightarrow\) \(\delta(f)\)

- The spectrum of a constant signal \( g(t) = 1 \) is an impulse \( \delta(f) = 2\pi \delta(2\pi f) \)

- \( g(t) = 1 \) is a dc signal that has frequency \( f = 0(\text{dc}) \)
Find the inverse Fourier transform of $\delta(f - f_0)$

- From sampling property of the impulse function

$$F^{-1}[\delta(f - f_0)] = \int_{-\infty}^{\infty} \delta(f - f_0) e^{j2\pi ft} df = e^{j2\pi f_0 t}$$

$e^{j2\pi f_0 t} \iff \delta(f - f_0)$

- This shows that the spectrum of an everlasting exponential $e^{j2\pi f_0 t}$ is a single impulse at $f = f_0$

- The spectrum is made up of a single component at frequency $f = f_0$

$$e^{-2\pi f_0 t} \iff \delta(f + f_0)$$
Find the Fourier transform of the everlasting sinusoid $\cos 2\pi f_0 t$

- Using Euler formula
  $$\cos 2\pi f_0 t = \frac{1}{2} (e^{j 2\pi f_0 t} + e^{-j 2\pi f_0 t})$$
  $$\cos 2\pi f_0 t = \frac{1}{2} [\delta(f + f_0) + \delta(f - f_0)]$$

- The spectrum of $\cos 2\pi f_0 t$ consists of two impulses at $f_0$ and $-f_0$ in the $f$-domain or two impulses $\pm \omega_0 = \pm 2\pi f_0$ in the $\omega$-domain.

- An everlasting sinusoid can be synthesized by two everlasting exponentials $e^{j \omega_0 t}$ and $e^{j \omega_0 t}$.

- The Fourier spectrum consists of only two components of frequencies $\omega_0$ and $-\omega_0$. 

\[\begin{align*}
\text{g(t)} & \quad \cos \omega_0 t \\
\text{t} & \quad \omega_0 \\
\delta(f + f_0) & \quad \delta(f - f_0) \\
G(\omega) & \quad \pi \quad \omega_0 \quad \omega_0 \\
\end{align*}\]
Find the Fourier transform of the sign function $\text{sgn}(t)$ – (signum $t$) its value is +1 or -1 depending on whether $t$ is positive or negative

$$\text{sgn } t = \begin{cases} 
1 & t > 0 \\
-1 & t < 0 
\end{cases}$$

$$\text{sgn } t = \lim_{a \to 0} \left[ e^{-at} u(t) - e^{at} u(-t) \right]$$

$$\mathcal{F} \left[ \text{sgn } t \right] = \lim_{a \to 0} \left\{ \mathcal{F}[e^{-at} u(t)] - \mathcal{F}[e^{at} u(-t)] \right\}$$

$$= \lim_{a \to 0} \left( \frac{1}{a + j\omega} - \frac{1}{a - j\omega} \right) = \frac{2}{j\omega}$$

(see pairs 1 and 2 in Table 3.1)
<table>
<thead>
<tr>
<th>$g(t)$</th>
<th>$G(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^{-at}u(t)$</td>
<td>$\frac{1}{a + j2\pi f}$</td>
</tr>
<tr>
<td>$a &gt; 0$</td>
<td></td>
</tr>
<tr>
<td>$e^{at}u(-t)$</td>
<td>$\frac{1}{a - j2\pi f}$</td>
</tr>
<tr>
<td>$a &gt; 0$</td>
<td></td>
</tr>
<tr>
<td>$e^{-a</td>
<td>t</td>
</tr>
<tr>
<td>$a &gt; 0$</td>
<td></td>
</tr>
<tr>
<td>$te^{-at}u(t)$</td>
<td>$\frac{1}{(a + j2\pi f)^2}$</td>
</tr>
<tr>
<td>$a &gt; 0$</td>
<td></td>
</tr>
<tr>
<td>$t^n e^{-at}u(t)$</td>
<td>$\frac{1}{(a + j2\pi f)^{n+1}}$</td>
</tr>
<tr>
<td>$a &gt; 0$</td>
<td></td>
</tr>
<tr>
<td>$\delta(t)$</td>
<td>$\delta(f)$</td>
</tr>
<tr>
<td>$1$</td>
<td></td>
</tr>
<tr>
<td>$e^{j2\pi f_0t}$</td>
<td>$\delta(f - f_0)$</td>
</tr>
<tr>
<td>$\cos 2\pi f_0t$</td>
<td>$0.5 [\delta(f + f_0) + \delta(f - f_0)]$</td>
</tr>
<tr>
<td>$\sin 2\pi f_0t$</td>
<td>$j0.5 [\delta(f + f_0) - \delta(f - f_0)]$</td>
</tr>
<tr>
<td>$u(t)$</td>
<td>$\frac{1}{2} \delta(f) + \frac{1}{j2\pi f}$</td>
</tr>
<tr>
<td>$\text{sgn } t$</td>
<td>$\frac{2}{j2\pi f}$</td>
</tr>
</tbody>
</table>
\begin{align*}
13 \quad \cos 2\pi f_0 t \ u(t) & \quad \frac{1}{4} \left[ \delta(f - f_0) + \delta(f + f_0) \right] + \frac{j 2\pi f}{(2\pi f_0)^2 - (2\pi f)^2} \\
14 \quad \sin 2\pi f_0 t \ u(t) & \quad \frac{1}{4j} \left[ \delta(f - f_0) - \delta(f + f_0) \right] + \frac{2\pi f_0}{(2\pi f_0)^2 - (2\pi f)^2} \\
15 \quad e^{-a t} \sin 2\pi f_0 t \ u(t) & \quad \frac{2\pi f_0}{(a + j2\pi f)^2 + 4\pi^2 f_0^2} \\
16 \quad e^{-a t} \cos 2\pi f_0 t \ u(t) & \quad \frac{a + j2\pi f}{(a + j2\pi f)^2 + 4\pi^2 f_0^2} \\
17 \quad \Pi \left( \frac{t}{\tau} \right) & \quad \tau \ \text{sinc} \left( \pi f \tau \right) \\
18 \quad 2B \ \text{sinc}(2\pi B t) & \quad \Pi \left( \frac{f}{2B} \right) \\
19 \quad \Delta \left( \frac{t}{\tau} \right) & \quad \frac{\tau}{2} \ \text{sinc}^2 \left( \frac{\pi f \tau}{2} \right) \\
20 \quad B \ \text{sinc}^2(\pi B t) & \quad \Delta \left( \frac{f}{2B} \right) \\
21 \quad \sum_{n=-\infty}^{\infty} \delta(t - nT) & \quad f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0) \\
22 \quad e^{-t^2/2\sigma^2} & \quad \sigma \sqrt{2\pi} e^{-2(\sigma^2 f)^2} \\
23 \quad f_0 = \frac{1}{T} \
\end{align*}
Some properties of Fourier Transform

- Properties of Fourier transform, implications and Applications will be studied
- Time-Frequency Duality- similar to photograph and its negative

If $g(t) \leftrightarrow G(f)$

- Time-shifting property
- Dual of the time-shifting property
- There is role reversal, and we guarantee that any result will have a dual
**Duality Property**

- If the Fourier transform of \( g(t) \) is \( G(f) \) then the Fourier transform of \( G(t) \) with \( f \) replaced by \( t \) is \( g(-f) \).

- \( g(-f) \) is the original time domain signal with \( t \) replaced by \( -f \).

\[
g(t) \leftrightarrow G(f) \]

\[
G(t) \leftrightarrow g(-f) \]

\[
g(t) = \int_{-\infty}^{\infty} G(x)e^{j2\pi xt} \, dx
\]

\[
g(-t) = \int_{-\infty}^{\infty} G(x)e^{-j2\pi xt} \, dx
\]
Apply duality property to the pair of figures below

- Figure

- \( g(t) \) is the same as \( G(f) \), and \( g(-f) \) is the same as \( g(t) \) with \( t \) replaced by \(-f\)

\[
\Pi \left( \frac{t}{\tau} \right) \iff \tau \text{sinc} \left( \pi f \tau \right)
\]

\[
\Pi \left( \frac{t}{\alpha} \right) \iff \alpha \text{sinc} \left( \pi f \alpha \right)
\]

- Substituting \( \tau = 2\pi\alpha \)
Time-Scaling Property

• If \( g(t) \iff G(f) \)

• For any real constant \( a \)

\[ g(at) \iff \frac{1}{|a|} G \left( \frac{f}{a} \right) \]

• Proof

\[ \mathcal{F}[g(at)] = \int_{-\infty}^{\infty} g(at)e^{-j2\pi ft} \, dt = \frac{1}{a} \int_{-\infty}^{\infty} g(x)e^{-j2\pi f\left(\frac{x}{a}\right)} \, dx = \frac{1}{a} G \left( \frac{f}{a} \right) \]

• If \( a < 0 \)

\[ g(at) \iff \frac{-1}{a} G \left( \frac{f}{a} \right) \]

• A time compression of a signal results in a spectral expansion and vice versa
Time scaling property

- Compression in time by a factor $a$ means that the signal is varying more rapidly by the same factor.
- To synthesize, frequencies of the signal must be increased by $a$ (frequency spectrum expanded by $a$).
- A signal $\cos 4\pi f_0 t$ is the same as the signal $\cos 2\pi f_0 t$ time compressed by a factor of 2.
Reciprocity of signal and its bandwidth

- Time-scaling property implies that if \( g(t) \) is wider, its spectrum is narrower and vice versa.
- Doubling the signal duration halves its bandwidth and vice versa.
- Bandwidth of a signal is inversely proportional to the signal duration or (width in seconds).
- E.g., bandwidth of previous figures.
Example

• Show that $g(-t) \iff G(-f)$

• Use this result and the fact that $e^{-at}u(t) \iff 1/(a+j2\pi f)$

\[
e^{at}u(-t) \iff \frac{1}{a-j2\pi f}
\]

\[
e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)
\]

\[
e^{-a|t|} \iff \frac{1}{a+j2\pi f} + \frac{1}{a-j2\pi f} = \frac{2a}{a^2 + (2\pi f)^2}
\]
Time-Shifting Property

- Delaying a signal by $t_0$ second does not change its amplitude spectrum, however the phase spectrum is changed by $-2\pi ft_0$

- If

\[ g(t) \leftrightarrow G(f) \]

\[ g(t - t_0) \leftrightarrow G(f)e^{-j2\pi ft_0} \]

- Proof:

\[ \mathcal{F}[g(t - t_0)] = \int_{-\infty}^{\infty} g(t - t_0)e^{-j2\pi ft} \, dt \]

If $t - t_0 = x$

\[ \mathcal{F}[g(t - t_0)] = \int_{-\infty}^{\infty} g(x)e^{-j2\pi f(x + t_0)} \, dx \]

\[ = e^{-j2\pi ft_0} \int_{-\infty}^{\infty} g(x)e^{-j2\pi fx} \, dx = G(f)e^{-j2\pi ft_0} \]
Time-shifting Property

• Time delay in a signal causes linear phase shift in its spectrum
• To achieve the same delay, higher frequency sinusoids must undergo proportionately larger phase shift

\[
\cos 2\pi f(t - t_0) = \cos (2\pi ft - 2\pi ft_0)
\]
Example

• Find the Fourier transform

\[ e^{-a|t-t_0|} \leftrightarrow \frac{2a}{a^2 + (2\pi f)^2} e^{-j2\pi f t_0} \]

• Delay causes a linear phase spectrum
Frequency-Shifting Property

• Multiplication of a signal by a factor $e^{j\pi f_0 t}$ shifts the spectrum of the signal by $f=f_0$

• If $g(t) \leftrightarrow G(f)$

Then

• Proof:

$$\mathcal{F}[g(t)e^{j2\pi f_0 t}] = \int_{-\infty}^{\infty} g(t)e^{j2\pi f_0 t} e^{-j2\pi ft} \, dt = \int_{-\infty}^{\infty} g(t)e^{-j(2\pi f - 2\pi f_0)t} \, dt = G(f - f_0)$$

• Changing $f$ to $-f_0$

$$g(t)e^{-j2\pi f_0 t} \leftrightarrow G(f + f_0)$$
• Because $e^{j\pi f_0 t}$ is not a real function in practise, frequency shifting in practice is achieved by multiplying $g(t)$ by a sinusoid.

\[ g(t) \cos 2\pi f_0 t = \frac{1}{2} \left[ g(t) e^{j2\pi f_0 t} + g(t) e^{-j2\pi f_0 t} \right] \]

\[ g(t) \cos 2\pi f_0 t \leftrightarrow \frac{1}{2} [G(f - f_0) + G(f + f_0)] \]

• Multiplication of $g(t)$ by a sinusoid of frequency $f_0$ shifts the spectrum $G(f)$ by $\pm f_0$.

• Multiplication of a sinusoid $\cos 2\pi f_0 t$ by $g(t)$ amounts to modulating the sinusoid amplitude. (amplitude modulation)

• $\cos 2\pi f_0 t$ is the carrier, $g(t)$ is the modulating signal and $\cos 2\pi f_0 t \ g(t)$ is the modulated signal (chap 4 and 5)
• To sketch a signal $g(t) \cos 2\pi f_0 t$

$$g(t) \cos 2\pi f_0 t = \begin{cases} g(t) & \text{when } \cos 2\pi f_0 t = 1 \\ -g(t) & \text{when } \cos 2\pi f_0 t = -1 \end{cases}$$

• $g(t) \cos 2\pi f_0 t$ touches $g(t)$ when the sinusoid $\cos 2\pi f_0 t$ is at its peaks and touches when $\cos 2\pi f_0 t$ is at its negative peaks

• $g(t)$ and $-g(t)$ therefore acts as envelopes for the signal $g(t) \cos 2\pi f_0 t$

• $g(t)$ and $-g(t)$ are mirror images of each other about the horizontal axis.
Shifting the phase spectrum of a modulated signal

- By using \( \cos(2\pi f_0 t + \theta) \) instead of \( \cos 2\pi f_0 t \)
- If signal \( g(t) \) is multiplied by \( \cos(2\pi f_0 t + \theta) \),

\[
g(t) \cos(2\pi f_0 t + \theta_0) \iff \frac{1}{2} \left[ G(f - f_0) e^{j\theta_0} + G(f + f_0) e^{-j\theta_0} \right]
\]

- If \( \theta_0 = -\pi/2 \)

\[
g(t) \sin 2\pi f_0 t \iff \frac{1}{2} \left[ G(f - f_0) e^{-j\pi/2} + G(f + f_0) e^{j\pi/2} \right]
\]
Example

- Find and sketch the Fourier transform of the modulated signal \( g(t)\cos 2\pi f_0 t \) in which \( g(t) \) is a rectangular pulse \( \Pi(t/T) \) as shown below:

- From pair, \( G(f) \)

\[
\Pi \left( \frac{t}{T} \right) \iff T \text{sinc} (\pi fT)
\]

- Spectrum is shown below

\[
g(t) \cos 2\pi f_0 t \iff \frac{1}{2} [G(f + f_0) + G(f - f_0)]
\]
Application of Modulation

• Modulation is used to shift signal spectra in this scenarios:
  – Interference will occur if signals occupying the same frequency band are transmitted over the same medium
  – Example, if radio stations broadcast audio signals, receiver will not be able to separate them
  – Radio stations could be assigned different carrier frequency.
  – Radio station transmit modulated signal, thus shifting the signal spectrum to allocated band.
  – Both modulation and demodulation utilizes spectra shifting
  – Transmitting several signals simultaneously over a channel by using different frequency bands is called Frequency division multiplexing
Application of Modulation

• Audio frequencies are so low (large wavelength) that it will require impractical large antennas for radiation.
• Shifting the spectrum to a higher frequency (smaller wavelength) by modulation will solve the problem
Bandpass Signal

- If \( gc(t) \) and \( gs(t) \) are low-pass signal, each with a bandwidth \( B \) Hz or \( 2\pi B \) rad/s
  - \( gc(t)\cos^2 \pi f_0 t \) and \( gc(t)\sin^2 \pi f_0 t \) are both bandpass signals occupying the same band each with bandwidth \( 2B \) Hz.
  - Linear combination of both signal is a bandpass signal occupying the same band as that of either signal (\( 2B \) Hz)
  - General bandpass signal
    \[
    g_{bp}(t) = gc(t)\cos^2 \pi f_0 t + gc(t)\sin^2 \pi f_0 t
    \]

- Magnitude spectra of individual signal is symmetrical about \( \pm f_0 \) but not of their sum

\[
\text{circumstance}
\]

- General bandpass \( g_{bp}(t) \) could be expressed as:
  \[
  g_{bp}(t) = E(t) \cos [2\pi f_0 t + \psi(t)]
  \]
Bandpass Signal

- $g_c(t)$ and $g_s(t)$ are low-pass, $E(t)$ and $\Psi(t)$ are also low-pass signals
Example

• Find the Fourier transform of a general periodic signal \( g(t) \) of period \( T_0 \) and hence determine the Fourier transform of the periodic impulse train \( \delta_{T_0}(t) \)

• A periodic signal \( g(t) \) can be expressed as an exponential Fourier series as

\[
g(i) = \sum_{n=-\infty}^{\infty} D_n e^{jn2\pi f_0 t} \quad f_0 = \frac{1}{T_0}
\]

\[
g(t) \iff \sum_{n=-\infty}^{\infty} F[D_n e^{jn2\pi f_0 t}]
\]

\[
g(t) \iff \sum_{n=-\infty}^{\infty} D_n \delta(f - nf_0)
\]
• $D_n = 1/T_0$

\[
\delta_{T_0}(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} e^{jn2\pi f_0 t} \quad f_0 = \frac{1}{T_0}
\]

\[
\delta_{T_0}(t) \iff \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta(f - nf_0)
\]

\[
= \frac{1}{T_0} \delta_{f_0}(f) \quad f_0 = \frac{1}{T_0}
\]

• Spectrum of the impulse train is an impulse train (frequency domain)
Convolution Theorem

- Convolution of two function \( g(t) \) and \( w(t) \)

\[
g(t) * w(t) = \int_{-\infty}^{\infty} g(\tau)w(t - \tau) \, d\tau
\]

- If

\[
g_1(t) \leftrightarrow G_1(f) \quad \text{and} \quad g_2(t) \leftrightarrow G_2(f)
\]

- Then time convolution

\[
g_1(t) * g_2(t) \leftrightarrow G_1(f)G_2(f)
\]

- Frequency convolution

\[
g_1(t)g_2(t) \leftrightarrow G_1(f) * G_2(f)
\]

- Time convolution of two signal in the time domain becomes multiplication in the frequency domain

- Multiplication of two signal in the time domain becomes convolution of the signals in the frequency domain
Convolution Theorem

• Prove

\[ \mathcal{F}[g_1(t) * g_2(t)] = \int_{-\infty}^{\infty} e^{-j2\pi ft} \left[ \int_{-\infty}^{\infty} g_1(\tau) g_2(t - \tau) d\tau \right] dt \]

\[ = \int_{-\infty}^{\infty} g_1(\tau) \left[ \int_{-\infty}^{\infty} e^{-j2\pi ft} g_2(t - \tau) dt \right] d\tau \]

• Using time shifting property

\[ \mathcal{F}[g_1(t) * g_2(t)] = \int_{-\infty}^{\infty} g_1(\tau) e^{-j2\pi f\tau} G_2(f) d\tau \]

\[ = G_2(f) \int_{-\infty}^{\infty} g_1(\tau)e^{-j2\pi f\tau} d\tau = G_1(f)G_2(f) \]

• Bandwidth of the product of two signals
  – If \( g_1(t) \) and \( g_2(t) \) have bandwidths \( B_1 \) and \( B_2 \), bandwidth of \( g_1(t) \)
    \( g_2(t) \) is \( B_1 + B_2 \)
  – Bandwidth of \( g^2(t) \) is \( 2B \) Hz and \( g^n(t) \) is \( nB \) Hz
Example

• Use the time convolution property to show

\[ g(t) \Leftrightarrow G(f) \]

Then

\[
\int_{-\infty}^{t} g(\tau) d\tau \Leftrightarrow \frac{G(f)}{j2\pi f} + \frac{1}{2} G(0) \delta(f)
\]

Knowing

\[
u(t - \tau) = \begin{cases} 
1 & \tau \leq t \\
0 & \tau > t
\end{cases}
\]

\[
g(t) * u(t) = \int_{-\infty}^{\infty} g(\tau)u(t - \tau) d\tau = \int_{-\infty}^{t} g(\tau) d\tau
\]

\[
g(t) * u(t) \Leftrightarrow G(f)U(f)
\]

\[
= G(f) \left[ \frac{1}{j2\pi f} + \frac{1}{2} \delta(f) \right]
\]

\[
= \frac{G(f)}{j2\pi f} + \frac{1}{2} G(0) \delta(f)
\]
Example

- Use the time differentiation property to find the Fourier transform of the triangular pulse $\Delta(t/\tau)$
- Involves successive differentiation as shown in b and c
- Second derivative consists of sequence of impulses
- Derivative of a signal at jump discontinuity is an impulse of strength equal to the amount of jump

$$\frac{d^2 g(t)}{dt^2} = \frac{2}{\tau} \left[ \delta\left(t + \frac{\tau}{2}\right) - 2\delta(t) + \delta\left(t - \frac{\tau}{2}\right) \right]$$

- Time differentiation property
- Time shifting property

\[
(j2\pi f)^2 G(f) = \frac{2}{\tau} \left( e^{j\pi f \tau} - 2 + e^{-j\pi f \tau} \right) = \frac{4}{\tau} \left( \cos \pi f \tau - 1 \right) = -\frac{8}{\tau} \sin^2 \left( \frac{\pi f \tau}{2} \right)
\]
Properties of Fourier Transform Operations

\[
G(f) = \frac{8}{(2\pi f)^2 \tau} \sin^2 \left( \frac{\pi f \tau}{2} \right) = \frac{\tau}{2} \left[ \frac{\sin (\pi f \tau/2)}{\pi f \tau/2} \right]^2 = \frac{\tau}{2} \operatorname{sinc}^2 \left( \frac{\pi f \tau}{2} \right)
\]

<table>
<thead>
<tr>
<th>Operation</th>
<th>(g(t))</th>
<th>(G(f))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Superposition</td>
<td>(g_1(t) + g_2(t))</td>
<td>(G_1(f) + G_2(f))</td>
</tr>
<tr>
<td>Scalar multiplication</td>
<td>(kg(t))</td>
<td>(kG(f))</td>
</tr>
<tr>
<td>Duality</td>
<td>(G(t))</td>
<td>(g(-f))</td>
</tr>
<tr>
<td>Time scaling</td>
<td>(g(at))</td>
<td>(\frac{1}{</td>
</tr>
<tr>
<td>Time shifting</td>
<td>(g(t - t_0))</td>
<td>(G(f) e^{-j2\pi ft_0})</td>
</tr>
<tr>
<td>Frequency shifting</td>
<td>(g(t)e^{j2\pi f_0 t})</td>
<td>(G(f - f_0))</td>
</tr>
<tr>
<td>Time convolution</td>
<td>(g_1(t) * g_2(t))</td>
<td>(G_1(f)G_2(f))</td>
</tr>
<tr>
<td>Frequency convolution</td>
<td>(g_1(t)g_2(t))</td>
<td>(G_1(f) * G_2(f))</td>
</tr>
<tr>
<td>Time differentiation</td>
<td>(\frac{d^n g(t)}{dt^n})</td>
<td>((j2\pi f)^n G(f))</td>
</tr>
<tr>
<td>Time integration</td>
<td>(\int_{-\infty}^{t} g(x) , dx)</td>
<td>(\frac{G(f)}{j2\pi f} + \frac{1}{2} G(0) \delta(f))</td>
</tr>
</tbody>
</table>
Signal transmission through a linear system

• Linear time-invariant (LTI) system can be characterized in the time domain or frequency domain
• The LTI system model can be used to characterize communication channels
• A stable LTI system can be characterized in the time domain by its impulse response \( h(t) \)
• Impulse response is the system response to a unit impulse input
• The system response to a bounded signal \( x(t) \) follows:
  \[
  y(t) = h(t) * x(t)
  \]
• Taking fourier transform \( Y(f) = H(f) \cdot X(f) \) (convolution theorem)
Analysis and Transmission of signal

• Fourier transform of the impulse response h(t), given as H(f) is called transfer function or frequency response of LTI system.

• H(f) is complex

\[ H(f) = |H(f)| e^{j\Theta h(f)} \]

|H(f)| is the amplitude response

\( \Theta h(f) \) is the phase response of the LTI system
Signal Distortion during Transmission

- During transmission, signal $x(t)$ is changed to signal $y(t)$
- If $X(f)$ and $Y(f)$ are spectra of the input and output respectively, $H(f)$ is the spectra response
  \[ Y(f) = H(f) \cdot X(f) \]
- Output spectrum is given by the input spectrum multiplied by the spectral response
- Equation above expressed in polar form
  \[ |Y(f)|e^{j\theta_y(f)} = |X(f)||H(f)|e^{j\theta_x(f)+\theta_h(f)} \]
- Amplitude and phase relationship
  \[ |Y(f)| = |X(f)||H(f)| \]
  \[ \theta_y(f) = \theta_x(f) + \theta_h(f) \]
- During transmission, input signal amplitude $|X(f)|$ is changed to $|X(f)| \cdot |H(f)|$
• Input signal phase spectrum $\theta x(f)$ is changed to $\theta x(f) + \theta h(f)$

• An input signal is modified by a factor of $|H(f)|$ and shifted in phase by $\theta h(f)$

• Plot of $|H(f)|$ and $\theta h(f)$ as a function of $f$ shows how the system modifies the amplitude and phase of various sinusoidal input

• $H(f)$ is the frequency response of the system

• During transmission some components are amplified and some are attenuated.

• Output will be different from input in general
Distortionless Transmission

- In applications such as message transmission over communication channel, the output waveform is required to be a replica of the input waveform.
- To achieve this, distortion due to amplification or communication channel must be minimized.
- Distortionless transmission is thus desired.
- Transmission is said to be distortionless if the input and the output have identical wave shapes within a multiplicative constant.
- A delayed output that retains the input waveform is distortionless.
- Given input $x(t)$ and output $y(t)$, a distortionless transmission satisfies:
  $$y(t) = k \cdot x(t - t_d)$$
Distortionless Transmission

• Fourier transform of previous equation:

\[ Y(f) = kX(f)e^{-j2\pi ft_d} \]

• Since \( Y(f) = X(f)H(f) \) then

\[ H(f) = k e^{-j2\pi ft_d} \]

• From previous equation

\[ |H(f)| = k \]
\[ \theta_h(f) = -2\pi ft_d \]

• For distortionless transmission, amplitude response \( |H(f)| \) must be a constant and phase response \( \theta_h(f) \) must be linear function of \( f \) going through the origin

• The slope of \( \theta_h(f) \) with respect to \( \omega = 2\pi f \) is \(-td\) (delay of output w.r.t input)
All-Pass vs. Distortionless System

- All-pass has constant gain for all frequencies (\(|H(f) = k|\) without linear phase requirement.
- Distortionless system is always and all-pass system but converse is not true.
- Transmitting recorded music signal that contains high frequency and low frequency component.
- An all-pass signal could cause extra delay on the high frequency component, which makes the music out of sync even if the signal components have the same gain and all components present.
- Difference in transmission delay is due to non-linear phase \(H(f)\) in the all-pass filter.
- For distortionless system \((t_d(f)\) have to be constant

\[ |H(f)| = k \quad \theta_h(f) = -2\pi ft_d \quad t_d(f) = -\frac{1}{2\pi} \cdot \frac{d\theta_h(f)}{df} \]
Distortion in Audio and Video Signal

- Human ear can readily perceive amplitude distortion but relatively insensitive to phase distortion
- To notice phase distortion, variation in delay (i.e in the slope of $\theta h$) should be comparable to signal duration
- In audio, each spoken syllable can be considered an individual signal.
- Audio systems may have nonlinear phases, yet no noticeable signal distortion because maximum variation in the slope of $\theta h$ is a small fraction of millisecond
- Audio manufacturers only provide $|H(f)|$
- In video, human eye is sensitive to the phase distortion but relatively insensitive to amplitude distortion
- In TV – smeared pictures
- In digital communication – pulse dispersion (spreading out)
Example

- If \( g(t) \) and \( y(t) \) are the input and the output respectively of a simple RC low-pass filter. Determine the transfer function \( H(f) \) and sketch \(|H(f)|, \theta h(f)\) and \( td(f) \). For distortionless transmission through this filter, what is the requirement on the bandwidth of \( g(t) \) if amplitude response variation within 2% and time delay variation within 5% are tolerable? What is the transmission delay? Find the output \( y(t) \).

- Applying voltage division rule

\[
H(f) = \frac{1/j2\pi fC}{R + (1/j2\pi fC)} = \frac{1}{1 + j2\pi fRC} = \frac{a}{a + j2\pi f}
\]

- Where

\[
a = \frac{1}{RC} = 10^6
\]
Example

- Hence

\[ |H(f)| = \frac{a}{\sqrt{a^2 + (2\pi f)^2}} \approx 1 \quad |2\pi f| \ll a \]

\[ \theta_h(f) = -\tan^{-1} \frac{2\pi f}{a} \approx -\frac{2\pi f}{a} \quad |2\pi f| \ll a \]

- Time delay

\[ t_d(f) = -\frac{d\theta_h}{d (2\pi f)} = \frac{a}{(2\pi f)^2 + a^2} \approx \frac{1}{a} = 10^{-6} \quad |2\pi f| \ll a \]

- The phase linearity results in a constant time delay characteristic. The filter can transmit low-frequency signals with negligible distortion
Example

• In our case, amplitude response within 2% and 5% is tolerable
• Let $f_0$ be the highest bandwidth of a signal that can be transmitted within these specification,
• To compute $f_0$, $|H(f)| = 1$ and $t_d(0) = 1/a$ second
• Hence $|H(f_0)| \geq 0.98$ and $t_d(f_0) \geq 0.95/a$

$$|H(f_0)| = \frac{a}{\sqrt{(2\pi f_0)^2 + a^2}} \geq 0.98 \implies 2\pi f_0 \leq 0.203a = 203,000 \text{ rad/s}$$

$$t_d(f_0) = \frac{a}{(2\pi f_0)^2 + a^2} \geq \frac{0.95}{a} \implies 2\pi f_0 \leq 0.2294a = 229,400 \text{ rad/s}$$
Ideal versus Practical filters

• Ideal filters allow distortionless transmission of a certain band of frequencies and suppress all remaining frequencies

• Ideal filter shown below allow all components below $f = B$ Hz to pass without distortion and suppresses all components above $f = B$

• If $g(t)$ is input signal and $y(t)$ is output then

$$y(t) = g(t - t_d)$$

• Ideal high pass and bandpass characteristic
Ideal versus Practical filters

- Signal $g(t)$ is transmitted without distortion but with delay $t_d$
- For this filter
  \[ |H(f)| = \Pi(f/2B), \text{ and } \theta(f) = -2\pi ft_d \]
  \[
  H(f) = \Pi \left( \frac{f}{2B} \right) e^{-j2\pi ft_d}
  \]
- Unit impulse response of this filter (using pair 18)
  \[
  h(t) = \mathcal{F}^{-1} \left[ \Pi \left( \frac{f}{2B} \right) e^{-j2\pi ft_d} \right]
  = 2B \text{sinc} [2\pi B(t - t_d)]
  \]
- For a physically realizable system $h(t)$ must be causal
  \[ h(t) = 0 \quad \text{for } t < 0 \]
- In frequency domain
  \[
  \int_{-\infty}^{\infty} \frac{|\ln|H(f)||}{1 + (2\pi f)^2} df < \infty
  \]
Ideal versus Practical filters

• The impulse response of previous filter is not realizable. To make it realizable (causal), the tail could be cut off

\[ \hat{h}(t) = h(t)u(t) \]

• Ideally a delay of \( t_d = \infty \) is needed for ideal filter

• The half-power bandwidth of a filter is defined as the bandwidth over which the amplitude response \(|H(f)|\) remains constant within a 3 dB or (ratio of 0.707)

• Half-power bandwidth of a lowpass filter is called the cutoff frequency
Digital Filter

• Analog signal can be processed by digital means (A/D conversion)
• Involves sampling, quantizing and coding
• Resulting digital signal can be processed by computer by algorithm e.g low-pass, high-pass filter algorithm

• Advantages of digital processing of analog signal
  – Computers can be timed shared, cost is usually lower, accuracy is dependent only on computer word length, quantizing interval and sampling rate (alias)
Signal Distortion over a Communication Channel

- Nature of signal distortion will be studied.
- Linear Distortion
  - Distortion from linear time invariant channel due to non-ideal characteristics of magnitude distortion, phase distortion or both
  - We can study the effect of the nonidealities on signal $g(t)$
  - Assuming the pulse exist within interval $(a, b)$ and is zero outside
  - The components of the Fourier spectrum of the pulse have a perfect and delicate balance of magnitudes and phases that adds up precisely to the pulse $g(t)$ over interval $[a, b]$
  - The balanced is left undisturbed if the signal is passed through a distortionless channel because a distortionless channel multiplies each component by the same factor and delays by the same time
Signal Distortion over a Communication Channel

– If the amplitude response of the channel is not ideal (i.e. \(|H(f)|\) is not equal to a constant), the balance will be disturbed, and the sum of all components cannot be zero outside the interval (a,b) – pulse will spread out.

– Dispersion also occurs if phase characteristic is not ideal

– Linear channel distortion (dispersion in time) is damaging to digital communication systems

– Leads to intersymbol interference (ISI) – digital symbol when transmitted over a dispersive channels tends to spread wider than its allotted time

– This makes adjacent symbols to interfere with one another thereby increasing the probability of detection error at the receiver
Example

• A low-pass filter transfer function $H(f)$ is given by

\[H(f) = \begin{cases} 
(1 + k \cos 2\pi f T) e^{-j2\pi ft_d} & |f| < B \\
0 & |f| > B 
\end{cases}\]

• A pulse $g(t)$ band-limited to $B$ Hz is applied at the input of this filter. Find the output $y(t)$.

• This filter has ideal phase and nonideal magnitude characteristics. Because $g(t) \Leftrightarrow G(f)$, $y(t) \Leftrightarrow Y(f)$ and

\[Y(f) = G(f)H(f) = G(f) \cdot \prod \left( \frac{f}{2B} \right) (1 + k \cos 2\pi f T) e^{-j2\pi ft_d}
\]

\[= G(f) e^{-j2\pi ft_d} + k \left[ G(f) \cos 2\pi f T \right] e^{-j2\pi ft_d}\]

• Because $g(t)$ is band-limited by to $B$ Hz

\[G(f) \cdot \prod \left( \frac{f}{2B} \right) = G(f)\]

\[y(t) = g(t-t_d) + \frac{k}{2} [g(t-t_d-T) + g(t-t_d+T)]\]
Distortion caused by Channel Nonlinearities

• Considering a memoryless nonlinear channel in which input \( g \) and the output \( y \) are related by some memoryless nonlinear equation

\[
y = f(g)
\]

• Expanding the right-hand side of this equation in a Maclaurin series:

\[
y(t) = a_0 + a_1 g(t) + a_2 g^2(t) + a_3 g^3(t) + \cdots + a_k g^k(t) + \cdots
\]

• Bandwidth of \( y(t) \) is greater than \( KB \) Hz hence the output spectrum spreads well beyond the input spectrum

• The output signal contains new frequency components not present in the input signal

• In broadcast communication, signals are amplified at high power levels where high efficiency amplifiers are desirable. However they (amplifiers) are nonlinear and they cause distortion. This is a problem in AM signals

• Non-linear distortion does not affect FM signal (chapter 5)
Distortion caused by channel nonlinearity

• If a signal is transmitted over a nonlinear channel, the nonlinearity not only distorts the signal, but also causes interference with other signals in the channel because of spectral dispersion

• Linear distortion causes interference between signals between the same channel

• Spectral distortion due to nonlinear distortion causes interference among signals using different frequency channels
Example

- The input $x(t)$ and the output $y(t)$ of a certain nonlinear channel are related as

$$y(t) = x(t) + 0.000158x^2(t)$$

- Find the output signal $y(t)$ and its spectrum $Y(f)$ if the input signal is $x(t) = 2000\text{sinc}(2000\pi t)$. Verify that the bandwidth of the output signal is twice that of the input signal. This is due to signal squaring. Can the signal $x(t)$ be recovered (without distortion) from the output $y(t)$.

$$x(t) = 2000 \text{sinc} (2000\pi t) \iff X(f) = \Pi \left( \frac{f}{2000} \right)$$

$$y(t) = x(t) + 0.000158x^2(t) = 2000 \text{sinc} (2000\pi t) + 0.316 \cdot 2000 \text{sinc}^2 (2000\pi t)$$

$$\iff$$

$$Y(f) = \Pi \left( \frac{f}{2000} \right) + 0.316 \Delta \left( \frac{f}{4000} \right)$$
Example

- $0.316 \cdot 2000 \text{sinc}^2(2000\pi t)$ is the unwanted (distortion) term in the received signal.

- The bandwidth of the received signal is twice that of the input signal $x(t)$ because of the squaring.

- The received signal contains the input signal $x(t)$ plus an unwanted signal $632 \text{sinc}^2(2000\pi t)$.

- Spectra of the desired signal and the distortion overlap, it is impossible to recover the signal $x(t)$ from the received signal $y(t)$ without distortion.

- Passing through a low pass filter gives (d) – still some residual distortion.
Distortion Caused by Multipath

- Multipath transmission occurs when transmitted signal arrives at the receiver by two or more paths of different delays due to impedance irregularities (mismatching).
- In radio links, the signal can be received by direct path between the transmitting and receiving antennas and also by reflections from other objects e.g. hills or building.

Transmission can be represented as several channel in parallel each with relative attenuation and different delay.

The transfer functions of the two paths are given by:

\[ e^{-j2\pi f t_d} \text{ and } \alpha e^{-j2\pi f (t_d + \Delta t)} \]
Distortion from Multipath

- The overall transfer function of the channel is \( H(f) \)

\[
H(f) = e^{-j2\pi ft_d} + \alpha e^{-j2\pi f(t_d+\Delta t)}
\]

\[
= e^{-j2\pi ft_d} (1 + \alpha e^{-j2\pi f \Delta t})
\]

\[
= e^{-j2\pi ft_d} (1 + \alpha \cos 2\pi f \Delta t - j\alpha \sin 2\pi f \Delta t)
\]

\[
= \sqrt{1 + \alpha^2 + 2\alpha \cos 2\pi f \Delta t} \exp \left[ -j \left( 2\pi ft_d + \tan^{-1} \frac{\alpha \sin 2\pi f \Delta t}{1 + \alpha \cos 2\pi f \Delta t} \right) \right]
\]

- Both magnitude and phase characteristics of \( H(f) \) are periodic in \( f \) with period of \( 1/\Delta t \) and can cause linear distortion

- If the gains from two paths are very close, then the signals received from the two paths may have opposite phase and will almost cancel each other
- E.g if \( f = \frac{n}{2\Delta t} (n = \text{odd}) \), \( \cos 2\pi f \Delta t = -1 \), \( H(f) = 0 \) when \( \alpha = 1 \). These frequencies are called multipath null frequencies.
- At frequencies \( f = \frac{n}{2\Delta t} (n = \text{even}) \), the two signal interfere constructively to enhance gain.
- This result in frequency selective fading of transmitted signals.
- In practice, channel characteristic vary over time due to random changes in the propagation medium.
- E.g. reflection properties vary with meteorological conditions.
- Effective channel transfer function varies semiperiodically and randomly causing random attenuation of the signal and this is called fading.
- Slow fading can be reduced by using Automatic gain control (AGC).
Fading Channel

- Fading could be strongly frequency dependent where different frequencies are affected unequally (Frequency selective fading)
Signal Energy and Energy Spectral Density

• Energy $E_g$ of a signal $g(t)$ is defined as the area under $|g(t)|^2$

• We can determine the energy signal from its Fourier transform $G(f)$ through Parseval’s theorem

\[
g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft} \, dt
\]

\[
E_g = \int_{-\infty}^{\infty} g(t)g^*(t) \, dt = \int_{-\infty}^{\infty} g(t) \left[ \int_{-\infty}^{\infty} G^*(f)e^{-j2\pi ft} \, df \right] \, dt
\]

\[
E_g = \int_{-\infty}^{\infty} G^*(f) \left[ \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft} \, dt \right] \, df
\]

\[
= \int_{-\infty}^{\infty} G(f)G^*(f) \, df
\]

\[
= \int_{-\infty}^{\infty} |G(f)|^2 \, df
\]

• Hence we can determine energy from both the time domain and the frequency domain
Example

• Verify the parseval’s theorem for the signal \( g(t) = e^{-at}u(t) \) \((a > 0)\)

\[
E_g = \int_{-\infty}^{\infty} g^2(t) \, dt = \int_{0}^{\infty} e^{-2at} \, dt = \frac{1}{2a}
\]

• Determining \( E_g \) from the signal spectrum

\[
G(f) = \frac{1}{j2\pi f + a}
\]

• Using parseval’s theorem

\[
E_g = \int_{-\infty}^{\infty} |G(f)|^2 \, df = \int_{-\infty}^{\infty} \frac{1}{(2\pi f)^2 + a^2} \, df = \frac{1}{2\pi a} \tan^{-1} \left( \frac{2\pi f}{a} \right) \bigg|_{-\infty}^{\infty} = \frac{1}{2a}
\]

• This verifies the Parseval theorem
Energy Spectral Density

• The previous example shows that the energy of a signal \( g(t) \) is the result of energies contributed by all the spectral components of the signal \( g(t) \).

• The contribution of a spectral component of frequency \( f \) is proportional to \( |G(f)| \)

• Assuming \( g(t) \) is input to a bandpass filter whose \( H(f) \) is

\[
Y(f) = G(f)H(f) \quad E_y = \int_{-\infty}^{\infty} |G(f)H(f)|^2 \, df
\]

\[
E_y = 2 |G(f_0)|^2 \, df
\]

• Energy spectral density (ESD) \( \Psi_g(t) \) (per unit bandwidth in Hz) is defined as

\[
\Psi_g(f) = |G(f)|^2 \quad E_g = \int_{-\infty}^{\infty} \Psi_g(f) \, df
\]

• The ESD of the signal \( g(t) = e^{-at}u(t) \)

\[
\Psi_g(f) = |G(f)|^2 = \frac{1}{(2\pi f)^2 + a^2}
\]
Essential Bandwidth of a signal

• Spectra of most signal extend to infinity
• However, because energy of most practical signal is finite, signal spectrum must approach 0 as f -> ∞
• Most of the signal energy is contained within certain band of B Hz and the energy content beyond B is negligible
• The bandwidth B is called essential bandwidth of the signal
• Criterion for selecting B depends on error tolerance in a particular application.
• You can select B to contain 95% of the signal energy
• Suppression of all spectral components of g(t) beyond the essential bandwidth results in \( \hat{g}(t) \) which is an approximation of g(t)
• If 95% criterion is used for essential bandwidth, energy of the error \( (g(t) - \hat{g}(t)) \) is 5% of Eg
Example

• Estimate the essential bandwidth $W$ (in rad/s) of the signal $g(t) = e^{-at}u(t)$ if the essential band is required to contain 95% of the signal

$$G(f) = \frac{1}{j2\pi f + a}$$

$$|G(f)|^2 = \frac{1}{(2\pi f)^2 + a^2}$$

• Signal energy $E_g$ is the area under ESD which $1/2a$. If $W$ rad/s is the essential bandwidth which contains 95% of the total energy $E_g$, 95% of the area in the figure

$$\frac{0.95}{2a} = \int_{-W/2\pi}^{W/2\pi} \frac{df}{(2\pi f)^2 + a^2}$$

$$= \frac{1}{2\pi a} \tan^{-1} \frac{2\pi f}{a} \bigg|_{-W/2\pi}^{W/2\pi} = \frac{1}{\pi a} \tan^{-1} \frac{W}{a}$$

$$\frac{0.95\pi}{2} = \tan^{-1} \frac{W}{a} \Rightarrow W = 12.7a \text{ rad/s}$$

$$B = \frac{W}{2\pi} = 2.02a \text{ Hz}$$

• or

• In Hz
Energy of Modulated Signal

- Modulation shifts the signal spectrum $G(f)$ to the left and right by $f_0$. Similar thing happen in ESD
- Let $g(t)$ be a baseband signal band-limited to $B$ Hz. The amplitude modulated signal is

$$\psi(t) = g(t) \cos 2\pi f_0 t$$

- And the spectrum of $\phi(t)$ is

$$\Phi(f) = \frac{1}{2} [G(f + f_0) + G(f - f_0)]$$

- The ESD of the modulated signal $\phi(t)$ is $|\Phi(f)|^2$ is

$$\Psi_\phi(f) = \frac{1}{4} |G(f + f_0) + G(f - f_0)|^2$$

- $F_0 \geq B$, then $G(f+f0)$ and $G(f-f0)$ are non overlapping

$$\Psi_\phi(f) = \frac{1}{4} \left[ |G(f + f_0)|^2 + |G(f - f_0)|^2 \right] = \frac{1}{4} \Psi_g (f + f_0) + \frac{1}{4} \Psi_g (f - f_0)$$

- Energy of modulated $E\Phi = \frac{1}{2}(Eg)$
Time autocorrelation Function and Energy Spectral Density

• Autocorrelation is an even function
  \[ \psi_g(\tau) = \psi_g(-\tau) \]

• Relationship between autocorrelation of a signal and its ESD. They form a fourier transform pair
  \[ \psi_g(\tau) \leftrightarrow \Psi_g(f) \]
  \[ \Psi_g(f) = \mathcal{F}\{\psi_g(\tau)\} = \int_{-\infty}^{\infty} \psi_g(\tau)e^{-j2\pi f \tau} d\tau \]
  \[ \psi_g(\tau) = \mathcal{F}^{-1}\{\Psi_g(f)\} = \int_{-\infty}^{\infty} \Psi_g(f)e^{+j2\pi f \tau} df \]

• Proof in textbook
  \[ \psi_g(\tau) \leftrightarrow \Psi_g(f) = |G(f)|^2 \]
ESD of the input and the Output

• If \( x(t) \) and \( y(t) \) are the input and the corresponding output of an LTI system then

\[
Y(f) = H(f)X(f)
\]

\[
|Y(f)|^2 = |H(f)|^2|X(f)|^2
\]

\[
\Psi_y(f) = |H(f)|^2\Psi_x(f)
\]

• The output signal ESD is \(|H(f)|^2\) times the input signal ESD
Signal Power and Power Spectral Density (PSD)

- Power is the area under PSD
- PSD is

\[
S_g(f) = \lim_{T \to \infty} \frac{|G_f(f)|^2}{T}
\]

\[
P_g = \int_{-\infty}^{\infty} S_g(f) \, df
\]

\[
= 2 \int_{0}^{\infty} S_g(f) \, df
\]